Supplement to:

Derivation of Equation (15)

Starting with Equation (14),

\[
\begin{align*}
\int_{-\infty}^{\infty} \int_{-\gamma Z/\sigma_u}^{\infty} ef(\varepsilon, u, \theta) d\varepsilon du & \quad - \quad \int_{-\infty}^{\infty} \int_{-\gamma Z/\sigma_u}^{\infty} ef(\varepsilon, u, \theta) d\varepsilon du \\
\int_{-\infty}^{\infty} \int_{-\gamma Z/\sigma_u}^{\infty} f(\varepsilon, u, \theta) d\varepsilon du & \quad - \quad \int_{-\infty}^{\infty} \int_{-\gamma Z/\sigma_u}^{\infty} f(\varepsilon, u, \theta) d\varepsilon du
\end{align*}
\]

the first term in the denominator can be written

\[
\int_{-\infty}^{\infty} \int_{-\gamma Z/\sigma_u}^{\infty} f(\varepsilon, u, \theta) d\varepsilon du = \int_{-\gamma Z/\sigma_u}^{\infty} f(u) du = \Pr \left[ u > -\gamma Z/\sigma_u \right] = 1 - F \left[ -\gamma Z/\sigma_u \right] = p(Z)
\]

and similarly for the other part of the denominator:

\[
\int_{-\infty}^{\infty} \int_{-\gamma Z/\sigma_u}^{\infty} f(\varepsilon, u, \theta) d\varepsilon du = 1 - p(Z)
\]

Then we have, for the whole expression,

\[
\begin{align*}
\int_{-\infty}^{\infty} \int_{-\gamma Z/\sigma_u}^{\infty} ef(\varepsilon, u, \theta) d\varepsilon du \times (1 - p(Z)) \quad - \quad \int_{-\infty}^{\infty} \int_{-\gamma Z/\sigma_u}^{\infty} ef(\varepsilon, u, \theta) d\varepsilon du \times p(Z)
\end{align*}
\]

\[
\frac{p(Z)(1 - p(Z))}{p(Z)(1 - p(Z))}
\]
and, because $E(\epsilon) = 0$ this reduces to

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\epsilon f(\epsilon, u, \theta)du d\epsilon}{p(Z)(1 - p(Z))} = \frac{\int_{-\infty}^{\infty} E(\epsilon|u)f(u)du}{(1 - p(Z))}$$

as required. In the case where $\epsilon$ and $u$ are bivariate normal, we can write the conditional expectation of $\epsilon$ given $u$ as

$$\rho_{\epsilon,u} \frac{\sigma_{\epsilon}}{\sigma_{u}} \left[ \frac{\phi \left( -\frac{\gamma Z}{\sigma_{u}} \right)}{\Phi \left( -\frac{\gamma Z}{\sigma_{u}} \right)} \right] = \rho_{\epsilon,u} \frac{\sigma_{\epsilon}}{\sigma_{u}} \left[ \frac{\phi \left( \Phi^{-1} \left( p(Z) \right) \right)}{p(Z)} \right]$$

Substituting this into the previous expression gives us Equation (13) (under the usual assumption that $u$ has a standard deviation of 1).